

An application of the MTP_2 property on bounds on system reliability

J. Gåsemyr and B. Natvig

Institute of Mathematics, University of Oslo, 0316 Oslo, Norway

This paper is concerned with the joint prior distribution of the dependent reliabilities of the components of a binary system. When this distribution is MTP_2 (Multivariate Totally Positive of Order 2), it is shown in general that this actually makes the machinery of Natvig and Eide [6] available to arrive at the posterior distribution of the system's reliability, based on data both at the component and system level. As an illustration in a common environmental stress case the joint prior distribution of the reliabilities is shown to have the MTP_2 property. We also show, similar to Gåsemyr and Natvig [2], for the case of independent components given component reliabilities how this joint prior distribution may be based on the combination of expert opinions. A specific system is finally treated numerically.

COMMON CAUSE FAILURES; COMBINED JUDGEMENT

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1. INTRODUCTION AND BASIC RESULTS

Consider, for a fixed point in time, t , a binary system of n binary components. Such a system has until now been used as a tool in reliability analyses of for instance nuclear power plants. Let $(i = 1, \dots, n)$:

$$\begin{aligned} X_i &= \begin{cases} 1 & \text{if the } i\text{th component functions} \\ 0 & \text{otherwise,} \end{cases} \\ \underline{X} &= (X_1, \dots, X_n), \\ \phi(\underline{X}) &= \begin{cases} 1 & \text{if the system functions} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Let furthermore:

$$\begin{aligned} E(X_i|p_i) &= p_i = \text{the reliability of the } i\text{th component,} \\ E(\phi(\underline{X})|h) &= h = \text{the reliability of the system.} \end{aligned}$$

If we assume that X_1, \dots, X_n are independent given $\underline{p} = (p_1, \dots, p_n)$, we write:

$$h = E(\phi(\underline{X})|\underline{p}) = h(\underline{p}).$$

Natvig and Eide [6] assumed that the joint prior distribution of the reliabilities, before running any experiments on the component level, $\pi(\underline{p})$, can be written as:

$$\pi(\underline{p}) = \prod_{i=1}^n \pi_i(p_i), \tag{1}$$

where $\pi_i(p_i)$ is the prior marginal distribution of p_i ; i.e., the components' individual reliabilities were deemed to be independent a priori. This may be unrealistic in an actual application. $\pi_i(p_i)$ describes the initial uncertainty in p_i , by for instance allocating most of the probability mass close to 1 indicating a very reliable component.

In the following we discuss the problem of deriving a distribution of system reliability based on prior information, $\pi(\underline{p})$, which is not necessarily in the form (1), as well as real data. Let us first consider the case of independent components given \underline{p} . Suppose that we run experiments on the component level and get the data $\underline{D} = (D_1, \dots, D_n)$ where D_i is the data from the experiment on the i th component. Let $\pi(\underline{D}|\underline{p})$ be the corresponding likelihood function. Hence the posterior distribution of the reliabilities, $\pi(\underline{p}|\underline{D})$, is given by:

$$\pi(\underline{p}|\underline{D}) = \frac{\pi(\underline{D}|\underline{p})\pi(\underline{p})}{\int \pi(\underline{D}|\underline{p})\pi(\underline{p})d\underline{p}}. \quad (2)$$

The corresponding distribution of system reliability $\pi\{h(\underline{p})|\underline{D}\}$ can in *principle* be arrived at by using the transformation formula for joint probability distributions; see Theorem 2 of Gåsemyr and Natvig [3]. The prior dependencies between p_1, \dots, p_n are not creating too much extra trouble here. Using expert opinions at the system level, $\pi\{h(\underline{p})|\underline{D}\}$ may then be updated to the prior distribution of system reliability $\pi_0\{h(\underline{p})|\underline{D}\}$. If we now finally run an experiment on the system level and get the data D , we end up with the posterior distribution of system reliability $\pi\{h(\underline{p})|\underline{D}, D\}$.

Let us next consider the case of associated components given \underline{p} as discussed in Barlow and Proschan [1]. This is the challenging case, since in real life systems, there typically exists nonnegative dependence between component states. Due to this unspecified assumption of dependence there is no way of establishing an exact expression for $\pi(h|\underline{D})$. Markov Chain Monte Carlo methods (such as the Gibbs sampler and the Metropolis-Hastings algorithm) can for instance only be used for specific joint distributions. In our case it is not even possible to arrive at exact expressions for its first r moments. The best one can do is to arrive at bounds on these moments. From (2) the marginal posterior distribution of p_i , $\pi(p_i|\underline{D})$, is given by:

$$\pi(p_i|\underline{D}) = \frac{\int \pi(\underline{D}|\underline{p})\pi(\underline{p})d(\cdot, \underline{p})}{\int \pi(\underline{D}|\underline{p})\pi(\underline{p})d\underline{p}}, \quad (3)$$

where $(\cdot, \underline{p}) = (p_1, \dots, p_{i-1}, \cdot, p_{i+1}, \dots, p_n)$. This leads to the moments up till order r ($i = 1, \dots, n; j = 1, \dots, r$):

$$E(p_i^j|\underline{D}) = \frac{\int p_i^j \pi(\underline{D}|\underline{p})\pi(\underline{p})d\underline{p}}{\int \pi(\underline{D}|\underline{p})\pi(\underline{p})d\underline{p}}. \quad (4)$$

From (4) by applying results of Natvig and Eide [6] we arrive at bounds on:

$$E(h^j|\underline{D}), \quad j = 1, \dots, r, \quad (5)$$

of $\pi(h|\underline{D})$.

However, most bounds in this paper are based on the assumption that p_1, \dots, p_n are independent given \underline{D} . Sufficient conditions for this are that the components have independent

prior reliabilities, as in the often unrealistic (1), and that D_1, \dots, D_n are independent given \underline{p} , which is reasonable if for instance different laboratories are used for different components. From (5) one may adjust a proper $\pi(h|\underline{D})$, which may be further updated to $\pi_0(h|\underline{D})$ and $\pi(h|\underline{D}, D)$ as in the case of independent components.

The lower bounds of Theorem 2.7 of Natvig and Eide [6] are given by

$$\max_{1 \leq s \leq p} \prod_{i \in P_s} E(p_i^j | \underline{D}) \leq E(h^j | \underline{D}), \quad j = 1, \dots, r, \quad (6)$$

where P_1, \dots, P_p are the minimal path sets corresponding to ϕ . The bounds are valid also under the weaker, and more realistic, assumption that p_1, \dots, p_n are associated given \underline{D} . Under the same assumption an upper bound on $E(h|\underline{D})$ is obtained by applying (6) for $j = 1$ on the dual system. This gives:

$$E(h|\underline{D}) \leq \min_{1 \leq s \leq k} \prod_{i \in K_s} E(p_i | \underline{D}), \quad (7)$$

where K_1, \dots, K_k are the minimal cut sets corresponding to ϕ .

The assumption that p_1, \dots, p_n are associated given \underline{D} , however, is hard to verify directly from the definition of association. Now recall that a random vector (Z_1, \dots, Z_n) is MTP_2 (Multivariate Totally Positive of Order 2) if and only if its density, $f(\underline{z})$, is MTP_2 ; i.e., if:

$$f(\underline{x} \vee \underline{y}) f(\underline{x} \wedge \underline{y}) \geq f(\underline{x}) f(\underline{y}),$$

where for $\underline{x} = (x_1, \dots, x_n), \underline{y} = (y_1, \dots, y_n)$

$$\begin{aligned} \underline{x} \vee \underline{y} &= (\max(x_1, y_1), \dots, \max(x_n, y_n)) \\ \underline{x} \wedge \underline{y} &= (\min(x_1, y_1), \dots, \min(x_n, y_n)). \end{aligned}$$

From (2) and Proposition 3.3 of Karlin and Rinott [4] $\pi(\underline{p}|\underline{D})$ is MTP_2 in \underline{p} if $\pi(\underline{D}|\underline{p})$ and $\pi(\underline{p})$ both are MTP_2 in \underline{p} . However, we do not know of any bounds in reliability based directly on the MTP_2 property. On the other hand according to Theorem 4.2 of Karlin and Rinott [4] the MTP_2 property of $\pi(\underline{p}|\underline{D})$ implies the association of p_1, \dots, p_n given \underline{D} , which is just the weaker assumption mentioned above. In the case where D_1, \dots, D_n are independent given \underline{p} we have:

$$\pi(\underline{D}|\underline{p}) = \prod_{i=1}^n \pi(D_i|p_i), \quad (8)$$

and it follows from Proposition 3.3 of Karlin and Rinott [4] that $\pi(\underline{D}|\underline{p})$ is MTP_2 in \underline{p} . Hence what remains to establish the association of p_1, \dots, p_n given \underline{D} , is to ensure that $\pi(\underline{p})$ in fact possess the MTP_2 property.

We conclude this section by considering the following special class of systems:

$$\phi(\underline{X}) = X_i \phi(1_i, \underline{X}), \quad i \in E \subset \{1, \dots, n\};$$

i.e. the i th component is in series with the rest of the system for $i \in E$. We then have the following upper bounds on $E(h^j|\underline{D})$, $j = 1, \dots, r$, which are generally valid:

$$\begin{aligned} E(h^j|\underline{D}) &= E[E(X_i \phi(1_i, \underline{X}) | h, \underline{p}, \underline{D})]^j \\ &\leq \min_{i \in E} E[E(X_i | h, \underline{p}, \underline{D})]^j = \min_{i \in E} E(p_i^j | \underline{D}), \quad j = 1, \dots, r. \end{aligned} \quad (9)$$

2. THE COMMON ENVIRONMENTAL STRESS CASE

In this section as an illustration we consider a special case of a multiplicative model treated in Lindqvist [5]. Suppose that the component reliabilities under normal conditions are u_1, \dots, u_n , respectively modelled as independent random quantities with values in $[0,1]$. Furthermore, suppose that a common environmental stress, modelled by the random quantity u_{n+1} also with values in $[0,1]$, affects the components in such a way that the reliability of the i th component is given as:

$$p_i = u_i u_{n+1}, \quad i = 1, \dots, n. \quad (10)$$

We assume that the quantities (u_1, \dots, u_{n+1}) are independent. Hence \underline{p} is associated since increasing functions of independent quantities are associated. Furthermore, we assume u_i to be beta-distributed with parameters (a_i, b_i) , where $a_i, b_i > 0$, $i = 1, \dots, n+1$. We then show that \underline{p} is actually MTP_2 .

From our assumptions:

$$\begin{aligned} \pi(\underline{p}) &= \prod_{i=1}^{n+1} \frac{\Gamma(a_i + b_i)}{\Gamma(a_i)\Gamma(b_i)} \int_0^1 I\left(\max_{1 \leq i \leq n} p_i \leq u_{n+1}\right) \\ &\quad \times u_{n+1}^{-n} u_{n+1}^{a_{n+1}-1} (1 - u_{n+1})^{b_{n+1}-1} \prod_{i=1}^n (p_i/u_{n+1})^{a_i-1} (1 - p_i/u_{n+1})^{b_i-1} du_{n+1} \\ &= \prod_{i=1}^{n+1} \frac{\Gamma(a_i + b_i)}{\Gamma(a_i)\Gamma(b_i)} \int_0^1 \prod_{i=1}^n p_i^{a_i-1} I\left(\max_{1 \leq i \leq n} p_i \leq w\right) w^{a_{n+1}-1+n-\sum_{i=1}^n (a_i+b_i)} \\ &\quad \times (1 - w)^{b_{n+1}-1} \prod_{i=1}^n (w - p_i)^{b_i-1} dw. \end{aligned} \quad (11)$$

We start by showing that the indicator function

$$I\left(\max_{1 \leq i \leq n} p_i \leq w\right)$$

is MTP_2 in the variables p_i , $i = 1, \dots, n$ and w . Suppose $x_i \leq y_i$, $i = 1, \dots, n+1$. For any subset A of $\{1, \dots, n\}$, we must show that:

$$\begin{aligned} &I\left(\max_{1 \leq i \leq n} y_i \leq y_{n+1}\right) I\left(\max_{1 \leq i \leq n} x_i \leq x_{n+1}\right) \\ &\geq I\left(\max_{i \in A, j \in A^c} \{x_i, y_j\} \leq y_{n+1}\right) I\left(\max_{i \in A^c, j \in A} \{x_i, y_j\} \leq x_{n+1}\right). \end{aligned}$$

If $\max_{1 \leq i \leq n} y_i$ is obtained for a $j \in A^c$, then:

$$I\left(\max_{i \in A, j \in A^c} \{x_i, y_j\} \leq y_{n+1}\right) = I\left(\max_{i \leq i \leq n} y_i \leq y_{n+1}\right).$$

Hence the inequality is now straightforward. If on the other hand $\max_{1 \leq i \leq n} y_i$ is obtained for a $j \in A$, then:

$$I\left(\max_{i \in A^c, j \in A} \{x_i, y_j\} \leq x_{n+1}\right) = I\left(\max_{i \leq i \leq n} y_i \leq x_{n+1}\right).$$

Since this factor is exceeded by both factors on the left hand side of the inequality, also this case is done.

The integrand of (11) is now MTP_2 in the variables p_i , $i = 1, \dots, n$ and w since a product of MTP_2 functions is again MTP_2 (see Proposition 3.3 of Karlin and Rinott [4]), and since $f(x, y) = h(x - y)$ is MTP_2 if $\log h(x)$ is concave (see Barlow and Proschan [1], page 76). Integrating with respect to w gives an MTP_2 density for \underline{p} by Proposition 3.2 of Karlin and Rinott [4].

The integration in (11) can be carried through in a simple way by assuming b_i , $i = 1, \dots, n + 1$ to be integers and $a_{n+1} + b_{n+1} - 1 < \sum_{i=1}^n a_i$. The latter condition ensures that the term w^{-1} does not occur. We then have:

$$\begin{aligned} \pi(\underline{p}) &= \prod_{i=1}^{n+1} \frac{\Gamma(a_i + b_i)}{\Gamma(a_i)\Gamma(b_i)} \prod_{i=1}^n p_i^{a_i-1} \int_{\max_{1 \leq i \leq n} p_i}^1 w^{a_{n+1}-1-\sum_{i=1}^n a_i} \\ &\quad \times \sum_{j_{n+1}=0}^{b_{n+1}-1} \binom{b_{n+1}-1}{j_{n+1}} (-w)^{j_{n+1}} \prod_{i=1}^n \sum_{j_i=0}^{b_i-1} \binom{b_i-1}{j_i} (-p_i/w)^{j_i} dw \\ &= \prod_{i=1}^{n+1} \frac{\Gamma(a_i + b_i)}{\Gamma(a_i)} \prod_{i=1}^n p_i^{a_i-1} \sum_{j_1=0}^{b_1-1} \dots \sum_{j_{n+1}=0}^{b_{n+1}-1} (-1)^{\sum_{i=1}^{n+1} j_i} \prod_{i=1}^n p_i^{j_i} \\ &\quad \times \frac{1}{a_{n+1} + j_{n+1} - \sum_{i=1}^n (a_i + j_i)} \left[1 - (\max_{1 \leq i \leq n} p_i)^{a_{n+1} + j_{n+1} - \sum_{i=1}^n (a_i + j_i)} \right] \prod_{i=1}^{n+1} \frac{1}{(b_i - 1 - j_i)! j_i!}. \end{aligned} \quad (12)$$

From (11) one easily establishes that:

$$E(p_i^j) = \frac{\Gamma(a_{n+1} + b_{n+1})\Gamma(a_{n+1} + j)\Gamma(a_i + b_i)\Gamma(a_i + j)}{\Gamma(a_{n+1})\Gamma(a_{n+1} + b_{n+1} + j)\Gamma(a_i)\Gamma(a_i + b_i + j)}. \quad (13)$$

Assume now D_1, \dots, D_n are independent given \underline{p} , where $D_i = (n_i, x_i)$, $i = 1, \dots, n$ with:

$$\pi(D_i | p_i) = \binom{n_i}{x_i} p_i^{x_i} (1 - p_i)^{n_i - x_i}, \quad i = 1, \dots, n.$$

Then the posterior distribution of \underline{p} is given by:

$$\pi(\underline{p} | \underline{D}) = K(n_1, \dots, n_n; x_1, \dots, x_n) \pi(\underline{p}) \prod_{i=1}^n \binom{n_i}{x_i} p_i^{x_i} (1 - p_i)^{n_i - x_i}, \quad (14)$$

where

$$\left[K(n_1, \dots, n_n; x_1, \dots, x_n) \right]^{-1} = \int \pi(\underline{p}) \prod_{i=1}^n \binom{n_i}{x_i} p_i^{x_i} (1 - p_i)^{n_i - x_i} d\underline{p}.$$

We then have:

$$E\left(\prod_{i=1}^n p_i^{j_i} | \underline{D}\right) = \frac{K(n_1, \dots, n_n; x_1, \dots, x_n)}{K(n_1 + j_1, \dots, n_n + j_n; x_1 + j_1, \dots, x_n + j_n)} \prod_{i=1}^n \prod_{k=1}^{j_i} (x_i + k) / (n_i + k). \quad (15)$$

To arrive at the posterior distribution of \underline{p} given by (14) and the corresponding moments given by (15) is in principle straightforward since from (12) and (14) we get:

$$\begin{aligned}
& [K(n_1, \dots, n_n; x_1, \dots, x_n)]^{-1} \\
&= \prod_{i=1}^{n+1} \frac{\Gamma(a_i + b_i)}{\Gamma(a_i)} \prod_{i=1}^n \binom{n_i}{x_i} \sum_{j_1=0}^{b_1-1} \dots \sum_{j_{n+1}=0}^{b_{n+1}-1} (-1)^{\sum_{i=1}^{n+1} j_i} \\
&\times \frac{1}{a_{n+1} + j_{n+1} - \sum_{i=1}^n (a_i + j_i)} \prod_{i=1}^{n+1} \frac{1}{(b_i - 1 - j_i)! j_i!} \left\{ \prod_{i=1}^n \frac{\Gamma(a_i + j_i + x_i) \Gamma(n_i - x_i + 1)}{\Gamma(a_i + j_i + n_i + 1)} \right. \\
&- \sum_{k=1}^n \sum_{k_1=0}^{n_1-x_1} \dots \sum_{k_n=0}^{n_n-x_n} (-1)^{\sum_{i=1}^n k_i} \prod_{i=1}^n \binom{n_i - x_i}{k_i} \int_{p_k=0}^1 \prod_{\substack{i=1 \\ i \neq k}}^n \int_{p_i=0}^{p_k} p_i^{a_i+j_i+x_i+k_i-1} dp_i \\
&\times p_k^{a_{n+1}+j_{n+1}-\sum_{\substack{i=1 \\ i \neq k}}^n (a_i+j_i)+x_k+k_k-1} dp_k \left. \right\} \\
&= \prod_{i=1}^{n+1} \frac{\Gamma(a_i + b_i)}{\Gamma(a_i)} \prod_{i=1}^n \binom{n_i}{x_i} \sum_{j_1=0}^{b_1-1} \dots \sum_{j_{n+1}=0}^{b_{n+1}-1} (-1)^{\sum_{i=1}^{n+1} j_i} \\
&\times \frac{1}{a_{n+1} + j_{n+1} - \sum_{i=1}^n (a_i + j_i)} \prod_{i=1}^{n+1} \frac{1}{(b_i - 1 - j_i)! j_i!} \left\{ \prod_{i=1}^n \frac{\Gamma(a_i + j_i + x_i) \Gamma(n_i - x_i + 1)}{\Gamma(a_i + j_i + n_i + 1)} \right. \\
&- \sum_{k=1}^n \sum_{k_1=0}^{n_1-x_1} \dots \sum_{k_n=0}^{n_n-x_n} (-1)^{\sum_{i=1}^n k_i} \prod_{i=1}^n \binom{n_i - x_i}{k_i} \prod_{\substack{i=1 \\ i \neq k}}^n \frac{1}{a_i + j_i + x_i + k_i} \\
&\times \frac{1}{a_{n+1} + j_{n+1} + \sum_{i=1}^n (x_i + k_i)} \left. \right\}. \tag{16}
\end{aligned}$$

For the special case $b_i = 1, i = 1, \dots, n+1; n_i - x_i = 1, i = 1, \dots, n$, this simplifies to:

$$\begin{aligned}
& [K(n_1, \dots, n_n; x_1, \dots, x_n)]^{-1} \\
&= \prod_{i=1}^{n+1} a_i \prod_{i=1}^n n_i \frac{1}{a_{n+1} - \sum_{i=1}^n a_i} \left\{ \prod_{i=1}^n \frac{1}{(a_i + n_i - 1)(a_i + n_i)} \right. \\
&- \sum_{k=1}^n \sum_{k_1=0}^1 \dots \sum_{k_{k-1}=0}^1 \sum_{k_{k+1}=0}^1 \dots \sum_{k_n=0}^1 (-1)^{\sum_{\substack{i=1 \\ i \neq k}}^n k_i} \\
&\times \prod_{\substack{i=1 \\ i \neq k}}^n \left(\frac{1}{a_i + x_i + k_i} \right) \frac{1}{(a_{n+1} + \sum_{i=1}^n x_i + \sum_{\substack{i=1 \\ i \neq k}}^n k_i)(a_{n+1} + 1 + \sum_{i=1}^n x_i + \sum_{\substack{i=1 \\ i \neq k}}^n k_i)} \left. \right\}. \tag{17}
\end{aligned}$$

This formula is needed in Section 4.

An alternative is to use a resampling method as suggested by Smith and Gelfand [7]. Then it is possible to treat situations where the extra conditions on (a_i, b_i) , $i = 1, \dots, n+1$ are not satisfied, and also to consider systems that are too complex for the implementation of the calculations above. Here we consider rejection sampling. Let:

$$L(\underline{p}) = \prod_{i=1}^n p_i^{x_i} (1 - p_i)^{n_i - x_i}$$

$$M = \max_{\underline{p}} L(\underline{p}) = \prod_{i=1}^n (x_i/n_i)^{x_i} (1 - x_i/n_i)^{n_i - x_i}.$$

The following procedure will then by repetition generate independent samples from the posterior distribution of \underline{p} :

1. Generate independently u_i from the beta distribution with parameters (a_i, b_i) , $i = 1, \dots, n+1$.
2. Compute \underline{p} from (10).
3. Generate u from the uniform $[0,1]$ distribution.
4. If $u \leq L(\underline{p})/M$, accept \underline{p} . Otherwise, repeat steps 1–4.

By increasing the sample size the posterior moments (4) can be determined with a desired precision.

Since for this common environmental stress case $\pi(\underline{p})$ possess the MTP_2 property, and (8) is satisfied, the lower bounds in (6) and the upper bound in (7) on the posterior system reliability moments are valid and can be calculated from (4).

For the case of independent components given \underline{p} , where system reliability $h(\underline{p})$ can be calculated, the rejection sampling approach can be applied to approximate the corresponding distribution of system reliability $\pi\{h(\underline{p})|\underline{D}\}$.

3. DETERMINATION OF THE JOINT PRIOR DISTRIBUTION OF THE RELIABILITIES BASED ON THE COMBINATION OF EXPERT OPINIONS

In this section we consider the case of independent components given \underline{p} . We assume that k experts will provide the information about the reliabilities of the components. From the assessments of the experts we then arrive at the joint prior distribution, $\pi(\underline{p})$, of the reliabilities. We suppose that the background information of the experts, corresponding to imaginary observation sets, is parallel to the alternative approach of Gåsemeyr and Natvig [2] in terms of realizations of m independent sets of independent random variables $(Z_{1l}, \dots, Z_{n+1,l})$, $l = 1, \dots, m$. Here Z_{il} has a binomial distribution $(1, u_i)$, $i = 1, \dots, n+1$; $l = 1, \dots, m$.

The decision maker (DM) assesses that the j th expert has access to information on the realizations of the Z_{il} 's for l with indices in the set A_j , having k_j elements, $j = 1, \dots, k$. We have:

$$\bigcup_{j=1}^k A_j = \{1, \dots, m\}.$$

We assume that the prior distributions of u_i , $i = 1, \dots, n+1$ both for the DM and the j th expert are independent beta distributions with parameters (c_i, d_i) for the DM and (c_{ji}, d_{ji}) for the j th expert, $j = 1, \dots, k$; $i = 1, \dots, n+1$. These distributions are typically noninformative, by for example setting all parameters equal to 1.

Now let z_{il} be the realization of Z_{il} , $i = 1, \dots, n+1$; $l = 1, \dots, m$ and let:

$$x_{ji} = \sum_{l \in A_j} z_{il} \quad x_i = \sum_{l=1}^m z_{il}, \quad j = 1, \dots, k; \quad i = 1, \dots, n+1. \quad (18)$$

Here, x_{ji} is the total number of survivals with survival probability u_i , corresponding to the information from the j th expert. x_i is similarly the total number of survivals with survival probability u_i , corresponding to the whole set of information. By Bayes' theorem the posterior distributions of u_i , $i = 1, \dots, n+1$ both for the DM and the j th expert are independent beta distributions with parameters $(c_i + x_i, d_i + m - x_i)$ for the DM and $(c_{ji} + x_{ji}, d_{ji} + k_j - x_{ji})$ for the j th expert, $j = 1, \dots, k$; $i = 1, \dots, n+1$.

Everything now fits in with the deductions in Section 2 by setting $a_i = c_i + x_i$ and $b_i = d_i + m - x_i$, leading to a joint prior distribution, $\pi(\underline{p})$, of the reliabilities being MTP₂. The x_i 's are estimated from the x_{ji} 's. We return to this after having dealt with the x_{ji} 's. These quantities are arrived at by asking the j th expert to assess the reliability of the i th component, p_{ji} , and the probability that all components function, p_j , $j = 1, \dots, k$; $i = 1, \dots, n$. Using (10) and denoting the density of the beta distribution with parameters (a, b) by $\beta(u; a, b)$, we get:

$$\begin{aligned} p_{ji} &= \int_0^1 \dots \int_0^1 u_i w \prod_{k=1}^n \beta(u_k; c_{jk} + x_{jk}, d_{jk} + k_j - x_{jk}) \\ &\quad \times \beta(w; c_{j,n+1} + x_{j,n+1}, d_{j,n+1} + k_j - x_{j,n+1}) du_1 \dots du_n dw, \\ &\quad j = 1, \dots, k; \quad i = 1, \dots, n, \\ p_j &= \int_0^1 \dots \int_0^1 \prod_{i=1}^n (u_i w) \prod_{i=1}^n \beta(u_i; c_{ji} + x_{ji}, d_{ji} + k_j - x_{ji}) \\ &\quad \times \beta(w; c_{j,n+1} + x_{j,n+1}, d_{j,n+1} + k_j - x_{j,n+1}) du_1 \dots du_n dw, \\ &\quad j = 1, \dots, k, \end{aligned} \quad (19)$$

having applied the assumption of independent components given \underline{p} in establishing the expression for p_j .

By integrating we end up with the following set of $n+1$ equations to determine the x_{ji} 's from the p_{ji} 's and p_j 's:

$$\begin{aligned} (c_{ji} + x_{ji}) &= p_{ji} (c_{ji} + d_{ji} + k_j) (c_{j,n+1} + d_{j,n+1} + k_j) / (c_{j,n+1} + x_{j,n+1}), \\ &\quad j = 1, \dots, k; \quad i = 1, \dots, n, \\ \prod_{i=1}^n (c_{ji} + x_{ji}) &= p_j \prod_{i=1}^n (c_{ji} + d_{ji} + k_j) (c_{j,n+1} + d_{j,n+1} + k_j + i - 1) / (c_{j,n+1} + x_{j,n+1} + i - 1), \\ &\quad j = 1, \dots, k. \end{aligned} \quad (20)$$

By multiplying the first n equations and dividing the last equation by the resulting product, we get the following equation to determine $x_{j,n+1}$:

$$\prod_{i=2}^n \left(1 + \frac{i-1}{c_{j,n+1} + x_{j,n+1}}\right) = \left(p_j / \prod_{i=1}^n p_{ji}\right) \prod_{i=2}^n \left(1 + \frac{i-1}{c_{j,n+1} + d_{j,n+1} + k_j}\right). \quad (21)$$

Inserting the solution into the first n equations, x_{ji} , $i = 1, \dots, n$ are determined. This procedure is repeated for $j = 1, \dots, k$ and hence all x_{ji} 's are arrived at.

The left hand side of (21) is obviously strictly decreasing in $x_{j,n+1}$. Since $x_{j,n+1} \leq k_j$ and $d_{j,n+1} > 0$, the left hand side of (21) is larger than the corresponding product on the right hand side. Hence a unique solution $x_{j,n+1}$ to (21) exists, due to the fact that $d_{j,n+1} > 0$, only if p_j is sufficiently larger than $\prod_{i=1}^n p_{ji}$. This can be considered as a claim of consistency of the j th expert. The fact that $p_j > \prod_{i=1}^n p_{ji}$ for the expressions in (19) follows from Theorem 2.5 of Natvig and Eide [6], by noting the remark at the bottom of page 324 of this paper.

We now return to the estimation of the x_i 's from the x_{ji} 's. Following Section 5 of Gåsemeyr and Natvig [2] there exists a disjoint partition B_g , $g = 1, \dots, q$ of the set $\{1, \dots, m\}$ and subsets C_j , $j = 1, \dots, k$ of the set $\{1, \dots, q\}$ such that we have the representation:

$$A_j = \bigcup_{g \in C_j} B_g, \quad j = 1, \dots, k. \quad (22)$$

Furthermore, introduce:

$$s_{gi} = \sum_{l \in B_g} z_{il}, \quad g = 1, \dots, q; \quad i = 1, \dots, n+1. \quad (23)$$

Then from (18), (22) and (23), we have:

$$\begin{aligned} x_{ji} &= \sum_{g \in C_j} s_{gi}, & j &= 1, \dots, k; \quad i = 1, \dots, n+1, \\ x_i &= \sum_{g=1}^q s_{gi}, & i &= 1, \dots, n+1. \end{aligned} \quad (24)$$

Hence the problem would be solved if the s_{gi} 's were known. This may be done as in Gåsemeyr and Natvig [2] by for fixed i to estimate s_{gi} for $q - k$ values of g . Then the remaining s_{gi} 's are determined from the first equation in (24). Here we suggest a short cut. Introduce:

$$\begin{aligned} G_g &= \{j | B_g \subset A_j\}, & g &= 1, \dots, q, \\ n_g &= \text{number of elements in } B_g, & g &= 1, \dots, q. \end{aligned} \quad (25)$$

Then parallel to (5.9) of Gåsemeyr and Natvig [2] the following estimate for s_{gi} is suggested:

$$\hat{s}_{gi} = n_g \sum_{j \in G_g} x_{ji} / \sum_{j \in G_g} k_j, \quad g = 1, \dots, q; \quad i = 1, \dots, n+1. \quad (26)$$

Hence the following estimate for x_i is arrived at:

$$\hat{x}_i = \sum_{g=1}^q \hat{s}_{gi}, \quad i = 1, \dots, n+1. \quad (27)$$

In principle the x_{ji} 's and the x_i 's should be integers. This may be obtained by approximations to the real values. However, since the data are imaginary and none integer values give mathematical meaning, we feel that the approximations are not necessary.

4. A SPECIFIC SYSTEM

Consider the case $n = 3$ and let:

$$\phi(\underline{X}) = X_1(X_2 \parallel X_3);$$

i.e. component 1 is in series with the rest of the system which consists of a parallel module of components 2 and 3. For simplicity assume $(a_i, b_i) = (4, 1)$, $i = 1, \dots, 4$, and $(n_i, x_i) = (10, 9)$, $i = 1, 2, 3$ in (14). By applying (17) we get:

$$\begin{aligned} [K(10, 10, 10; 9, 9, 9)]^{-1} &= \\ &= 32000 \left\{ 3 \left[\frac{1}{13^2 \cdot 31 \cdot 32} - \frac{2}{13 \cdot 14 \cdot 32 \cdot 33} + \frac{1}{14^2 \cdot 33 \cdot 34} \right] - \frac{1}{13^3 \cdot 14^3} \right\} \\ &= 0.00485756 \end{aligned}$$

$$\begin{aligned} [K(11, 10, 10; 10, 9, 9)]^{-1} &= \\ &= 35200 \left\{ \frac{1}{13^2 \cdot 32 \cdot 33} - \frac{2}{13 \cdot 14 \cdot 33 \cdot 34} + \frac{1}{14^2 \cdot 34 \cdot 35} + 2 \left[\frac{1}{13 \cdot 14 \cdot 32 \cdot 33} \right. \right. \\ &\quad \left. \left. - \frac{1}{13 \cdot 15 \cdot 33 \cdot 34} - \frac{1}{14^2 \cdot 33 \cdot 34} + \frac{1}{14 \cdot 15 \cdot 34 \cdot 35} \right] - \frac{1}{13^2 \cdot 14^3 \cdot 15} \right\} = 0.0044578 \end{aligned}$$

$$\begin{aligned} [K(12, 10, 10; 11, 9, 9)]^{-1} &= \\ &= 38400 \left\{ \frac{1}{13^2 \cdot 33 \cdot 34} - \frac{2}{13 \cdot 14 \cdot 34 \cdot 35} + \frac{1}{14^2 \cdot 35 \cdot 36} + 2 \left[\frac{1}{13 \cdot 15 \cdot 33 \cdot 34} \right. \right. \\ &\quad \left. \left. - \frac{1}{13 \cdot 16 \cdot 34 \cdot 35} - \frac{1}{14 \cdot 15 \cdot 34 \cdot 35} + \frac{1}{14 \cdot 16 \cdot 35 \cdot 36} \right] - \frac{1}{13^2 \cdot 14^2 \cdot 15 \cdot 16} \right\} = 0.00409896. \end{aligned}$$

From (15) we now find:

$$E(p_i | \underline{D}) = \frac{0.0044578}{0.00485756} \cdot \frac{10}{11} = 0.834276, \quad i = 1, 2, 3,$$

$$E(p_i^2 | \underline{D}) = \frac{0.00409896}{0.00485756} \cdot \frac{5}{6} = 0.703193, \quad i = 1, 2, 3.$$

Note that for this special system and the information at hand, the upper bounds in (7) and in (9) for $j = 1$ are identical. Hence, by using (6) and (9) for $j = 1, 2$, we arrive at:

$$0.696016 = (0.834276)^2 \leq E(h | \underline{D}) \leq 0.834276 \quad (28)$$

$$0.49448 = (0.703193)^2 \leq E(h^2 | \underline{D}) \leq 0.703193.$$

Remember from Section 1 that due to the unspecified assumption of associated components given \underline{p} , it is not possible to arrive at exact expressions for $E(h^j|\underline{D})$, $j = 1, 2$. This is, however, possible if components are actually independent given \underline{p} . We then have:

$$h(\underline{p}) = p_1(p_2 + p_3 - p_2p_3).$$

By first applying (14) and (15) we get after finally using (17):

$$\begin{aligned} E(h|\underline{D}) &= E(p_1p_2 + p_1p_3 - p_1p_2p_3|\underline{D}) \\ &= E[2p_1p_2 - p_1p_2p_3|\underline{D}] \\ &= K(10, 10, 10; 9, 9, 9) \left\{ 2[K(11, 11, 10; 10, 10, 9)]^{-1} \left(\frac{10}{11}\right)^2 \right. \\ &\quad \left. - [K(11, 11, 11; 10, 10, 10)]^{-1} \left(\frac{10}{11}\right)^3 \right\} \\ &= K(10, 10, 10; 9, 9, 9) 32000 \left\{ 4 \left[\frac{1}{13 \cdot 14 \cdot 33 \cdot 34} - \frac{1}{13 \cdot 15 \cdot 34 \cdot 35} \right. \right. \\ &\quad \left. - \frac{1}{14^2 \cdot 34 \cdot 35} + \frac{1}{14 \cdot 15 \cdot 35 \cdot 36} \right] + 2 \left[\frac{1}{14^2 \cdot 33 \cdot 34} - \frac{2}{14 \cdot 15 \cdot 34 \cdot 35} + \frac{1}{15^2 \cdot 35 \cdot 36} \right] \\ &\quad \left. - \frac{2}{13 \cdot 14^3 \cdot 15^2} - 3 \left[\frac{1}{14^2 \cdot 34 \cdot 35} - \frac{2}{14 \cdot 15 \cdot 35 \cdot 36} + \frac{1}{15^2 \cdot 36 \cdot 37} \right] + \frac{1}{14^3 \cdot 15^3} \right\} \\ &= 0.81076 \end{aligned}$$

$$\begin{aligned} E(h^2|\underline{D}) &= E(p_1^2(p_2 + p_3 - p_2p_3)^2|\underline{D}) \\ &= E(2p_1^2p_2^2 + p_1^2p_2^2p_3^2 + 2p_1^2p_2p_3 - 4p_1^2p_2^2p_3|\underline{D}) \\ &= K(10, 10, 10; 9, 9, 9) \left\{ 2[K(12, 12, 10; 11, 11, 9)]^{-1} \left(\frac{10}{12}\right)^2 \right. \\ &\quad \left. + [K(12, 12, 12; 11, 11, 11)]^{-1} \left(\frac{10}{12}\right)^3 \right. \\ &\quad \left. + 2[K(12, 11, 11; 11, 10, 10)]^{-1} \left(\frac{10}{12}\right) \left(\frac{10}{11}\right)^2 - 4[K(12, 12, 11; 11, 11, 10)]^{-1} \left(\frac{10}{12}\right)^2 \left(\frac{10}{11}\right) \right\} \\ &= K(10, 10, 10; 9, 9, 9) 32000 \left\{ 4 \left[\frac{1}{13 \cdot 15 \cdot 35 \cdot 36} - \frac{1}{14 \cdot 15 \cdot 36 \cdot 37} \right. \right. \\ &\quad \left. - \frac{1}{13 \cdot 16 \cdot 36 \cdot 37} + \frac{1}{14 \cdot 16 \cdot 37 \cdot 38} \right] + 2 \left[\frac{1}{15^2 \cdot 35 \cdot 36} - \frac{2}{15 \cdot 16 \cdot 36 \cdot 37} + \frac{1}{16^2 \cdot 37 \cdot 38} \right] \\ &\quad \left. - \frac{2}{13 \cdot 14 \cdot 15^2 \cdot 16^2} + 3 \left[\frac{1}{15^2 \cdot 37 \cdot 38} - \frac{2}{15 \cdot 16 \cdot 38 \cdot 39} + \frac{1}{16^2 \cdot 39 \cdot 40} \right] - \frac{1}{15^3 \cdot 16^3} \right. \\ &\quad \left. + 2 \left[\frac{1}{14^2 \cdot 35 \cdot 36} - \frac{2}{14 \cdot 15 \cdot 36 \cdot 37} + \frac{1}{15^2 \cdot 37 \cdot 38} \right] + 4 \left[\frac{1}{14 \cdot 15 \cdot 35 \cdot 36} \right. \right. \\ &\quad \left. - \frac{1}{14 \cdot 16 \cdot 36 \cdot 37} - \frac{1}{15^2 \cdot 36 \cdot 37} + \frac{1}{15 \cdot 16 \cdot 37 \cdot 38} \right] - \frac{2}{14^2 \cdot 15^3 \cdot 16} \\ &\quad \left. - 8 \left[\frac{1}{14 \cdot 15 \cdot 36 \cdot 37} - \frac{1}{14 \cdot 16 \cdot 37 \cdot 38} - \frac{1}{15^2 \cdot 37 \cdot 38} + \frac{1}{15 \cdot 16 \cdot 38 \cdot 39} \right] \right. \\ &\quad \left. - 4 \left[\frac{1}{15^2 \cdot 36 \cdot 37} - \frac{2}{15 \cdot 16 \cdot 37 \cdot 38} + \frac{1}{16^2 \cdot 38 \cdot 39} \right] + \frac{4}{14 \cdot 15^3 \cdot 16^2} \right\} \\ &= 0.665288. \end{aligned}$$

Comparing the exact expressions for $E(h^j|\underline{D})$, $j = 1, 2$, for the case of independent components given \underline{p} , with the bounds in (28) we see that in this case the upper bounds are the better ones. The reason is that the parallel module of components 2 and 3 is very reliable, and can be replaced by a perfect module without affecting system reliability very much.

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